A Brief Report on

Ancient Mechanics

—Catapults

By

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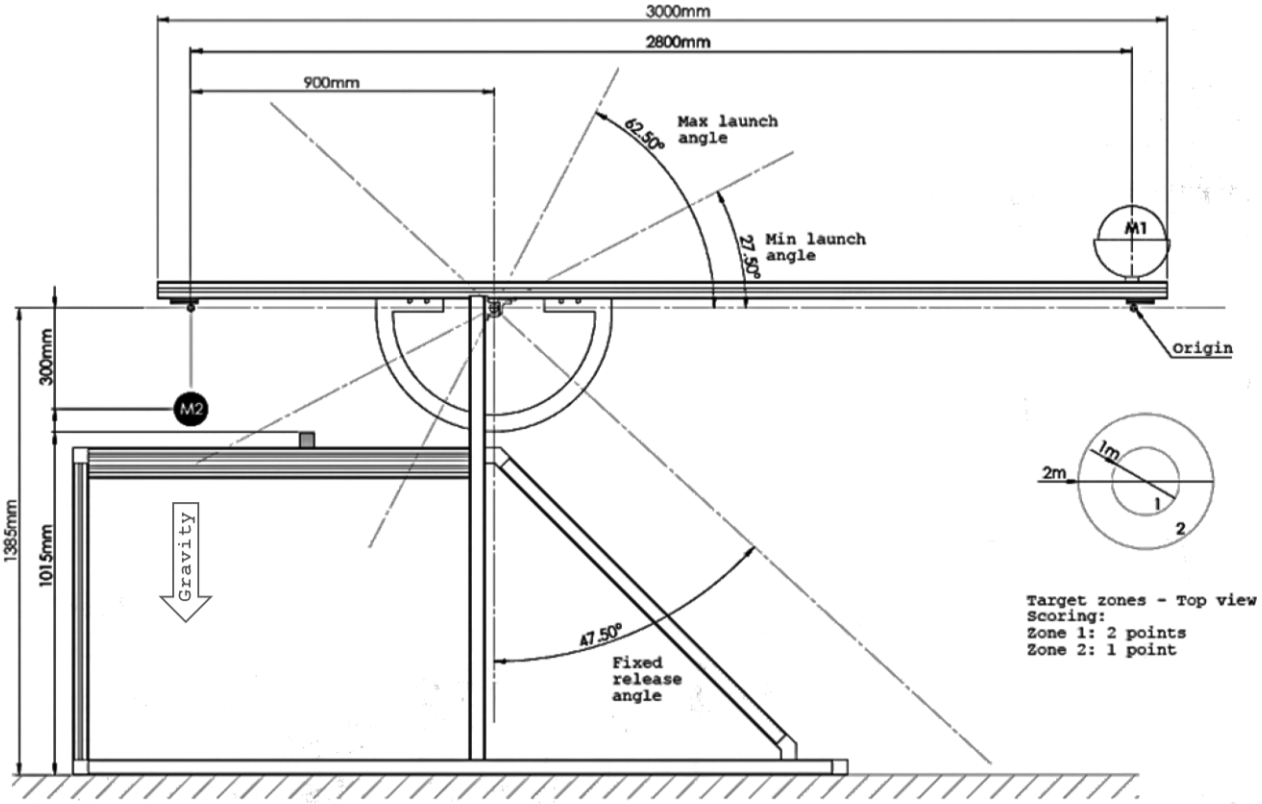
Aims:

Problem solving skills; report writing skills; data presentation skills; critical thinking skills and team working skills.

Introduction:



As a group of five we were given the measurements of a trebuchet catapult (roughly speaking a device that converts gravitational potential energy of its counter mass into the translational kinetic energy of a projectile) which was built by the Dyson School of Design Engineering staff. We were also given the diameter and mass of a spherical projectile and the distance of a target (castle!). We were challenged to analytically calculate for the angle of projection and the value of the counter mass required to hit the target accounting for any hypothetically influential factors (as well as to dress up as barbaric medieval people). Therefore, I dressed up as Mr. Donald Trump, and the calculations are as follows.

Calculations:

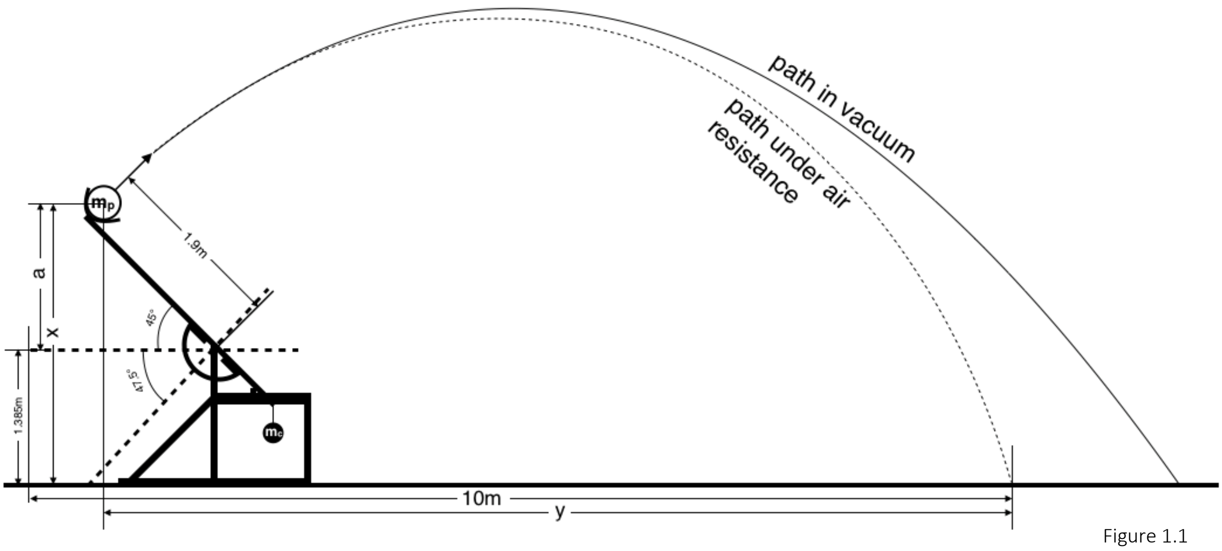


Figure 1.1

Parabolic Motion of Projectile:

(We chose 45° as our angle of projection for simpler calculations)

Height of the projectile when it leaves the beam:

$$x = 1.385 + 1.9\times \sin 45° = 2.7285(m)$$

Horizontal displacement of the projectile:

$$y = 10 - (1.9 - 1.9\times \cos 45°) = 9.4435(m)$$

First, we assume 0 air resistance to find an estimated v_0, this will be smaller than the actual v_0 but since $\bar{v}$ is what's required for the closest estimation of air resistance and $\bar{v} < v_0$, this is a reasonable estimation.

Vertical component of the parabolic motion:

$$y = v_{y0}t + \frac{1}{2}at^2 = -2.7285(m)$$

Horizontal component of the parabolic motion:

$$x = v_x t = 9.4435(m) \qquad v_x = \frac{9.4435}{t}(ms^{-1})$$

$$\because v_{x0} = \frac{x}{t} = v_{y0}$$

$$\therefore y = \frac{x}{t}t - \frac{1}{2}gt^2 = x - \frac{1}{2}gt^2$$

$$-2.7285 = 9.4435 - \frac{1}{2}9.81t^2$$

$$t = \sqrt{2.48154944} = 1.575(s)$$

because launching angle of the projectile is 45°:

$$v_0 = \frac{v_{y0}}{cos45°} = \frac{v_{x0}}{cos45°} = \frac{\frac{x}{t}}{cos45°} = \frac{\frac{9.4435}{1.575}}{cos45°} = 8.479(ms^{-1})$$

Air Resistance:

This is only an estimation of air resistance as the actual drag of an object cannot be calculated analytically but only by numerical simulations; something I found out the hard way after spending four hours on deriving the two differential equations and trying to solve it. Moreover, we are only considering aerodynamic drag and kinematic viscosity since we were not given the texture of the projectile and the force due to skin friction on a spherical object is on average around 10 times smaller than that caused by its aerodynamic drag[1]. Furthermore, we are neglecting the buoyancy of air since it will only produce a minimal effect on the result.

Equation for aerodynamic drag:

$$F_D = \frac{1}{2}\rho v^2 C_D A$$

Where:

$$\rho_{(air\ at\ 8℃\ and\ 1atm)} = 1.256(\mathrm{kg/m^3})$$

$C_{D(sphere)} = 0.47$[4]

$$A = \pi r^2 = 0.0284(m^2)$$

$$\therefore F_D = \frac{1}{2}1.256 \times 8.479^2 \times 0.47 \times 0.0284 = 0.588(N)$$

Equation for Kinematic Viscosity:

$$F_\mu = 6\pi\mu R v$$

Where:

$\mu_{(air)} = 1.983\times10^{-2}$

$R = 0.095(\mathrm{m})$

$\therefore F_\mu = 6\pi\times0.01983\times0.095\times8.479 = 0.301(N)$

Altogether this will provide a deceleration:

$$-a = \frac{F_D + F_\mu}{m} = \frac{0.588 + 0.301}{0.326} = 2.727ms^{-2}$$

Now we go back to the horizontal projectile motion:

$$x = v_{x0}t + \frac{1}{2}at^2$$

Since both drag and viscosity acts in the opposite direction of the projectile's velocity, and the projectile moves in both the positive and negative direction on the vertical axis in a single parabola to almost the same amount, Air resistance will have minimal effect on the vertical part of the parabolic motion. Therefore, we can assume that time t is unchanged.

$$9.4435(m) = v_{x0}\times1.575 - \frac{1}{2}\times2.727\times1.575^2$$

$$v_{x0} = 8.143(ms^{-1})$$

$$v_0 = \frac{v_{x0}}{cos45} = 11.516(ms^{-1})$$

Energy transformation:

G.P.E.Countermass →

G.P.E.beam, Rotational K.E.beam, Translational K.E.beam

G.P.E.projectile and Translational K.E.Projectile

$$m_C g\Delta h_c = m_b g\Delta h_b + \frac{1}{2}I_b\omega^2 + \frac{1}{2}m{v_b}^2 + m_p g\Delta h_p + \frac{1}{2}m_p{v_0}^2$$

Where:

$$\omega = \frac{v_0}{r} = \frac{11.516}{1.9} = 6.061(\theta s^{-1})$$

$$\Delta h_c = 0.9 \times sin45° + 0.9 \times sin47.5° = 1.300(m)$$

$$\Delta h_b = 0.5 \times sin45° + 0.5 \times sin47.5° = 0.722(m)$$

$$\Delta h_p = 1.9 \times sin45° + 1.9 \times sin47.5° = 2.744(m)$$

$$I_b = 5.680(kgm^2)$$

$$m_b = 7.500(kg)$$

$$v_b = \omega r = 6.061 \times 0.5 = 3.031(ms^{-1})$$

$$\therefore \boldsymbol{m_C} = (\boldsymbol{7.5 \times 9.81 \times 0.722 + \frac{1}{2} \times 5.68 \times 6.061^2 + \frac{1}{2} \times 7.5 \times 3.031^2 + 0.326 \times 9.81 \times 2.744 + \frac{1}{2} \times 0.326 \times 11.516^2) / 9.81 \times 1.3 = 17.4}(\boldsymbol{kg})$$

Results and Analysis:

Date: 11st of November, 2016
Time: 1:42pm

Experimental Results:

(We made a small mistake for the group calculation where instead of the cross-sectional area πr^2 we used $4\pi r^2$ for the projectile in the calculation for drag which increased the counter mass required from 17.4(kg) to 21.3(kg), which happened to be just right)

Trial No.	Projectile Mass (m_p/kg)	Projectile Radius (r/m)	Angle of Projection	Counter Mass Used (m_c/kg)	Projectile Displacement Relative to The Origin (x/m)
1	0.326	0.095	45°	21.3	10.0
2	0.326	0.095	45°	21.3	10.3
3	0.326	0.095	45°	21.3	10.6

Table 2.1

Other Independent Variables:

- Temperature: 8°C
- Average wind direction between 1 and 2pm: 191.25° - 213.75° clockwise from north
- Average wind speed between 1 and 2pm: 1.6ms^{-1}
- Direction of projectile: ~170° clockwise from north
- Humidity: 65%

Result Analysis:

Mean ($\bar{x}$/m)	Median (/m)	Mode (/m)	Range (/m)
10.3	10.3	-	0.6

Table 2.2

As shown in diagram 2.1, our group's first shot was a dead centre at 10.0 meters. The reason we did not change the counter mass was that we wanted to calculate for the range of error. However, the two following shots resulted in a linear shift towards the far side of the target by +0.3meters each time, this was unexpected.

One possible factor contributing to the error in the results may be the wind. This is because the general direction of the wind on the day was roughly along the direction of projection, this may have helped to push the relatively light ball further over the target. However, this does not help to explain the linearity in the shift.

A better hypothesis which can explain the linear increase in the projectile's displacement in the x direction is a gradual change in the angle of projection. My hypothesis is that as the beam of the trebuchet hits the bar used to control the angle of projection, the bar slid backwards by a tiny amount which increased the angle of projection (labelled 45° in figure 1.1). This in turn reduced the initial tangent of the parabola. Since the optimum initial tangent to a parabolic motion under air resistance is small than 45°[2], this explains the linear increase in the displacement x that was observed.

Sources of Error in the Calculation:

Assuming that m_c=21.3kg and θ=45° were correct values, and that θ=45° is constant:

Experimental value of m_c/kg	Analytical value of m_c/kg	Difference in Experimental and Analytical values of m_c/kg	Percentage error (%)
21.3	17.4	3.9	18.3

Table 3.1

As given in the diagram above, the analytical value for the counter mass was by magnitude 18.3% smaller than that of the experimental value. This implies that there must be errors caused by the simplifications made in the calculations and/or some additional loss of energy.

Errors from simplification:

There are two main sources of errors caused by simplifications used in the calculations. Firstly, we assumed that the surface of the projectile was smooth and therefore neglected air resistance caused by skin friction. In reality however, we were given a relatively rough ball which might have significantly increased the drag coefficient C_D.

Another considerable source of error caused by oversimplification is in the initial velocity v_0 that we plugged into the drag equation. In this calculation, we used the initial velocity required to launch the projectile into the target at 0 air resistance, which is significantly smaller than the real value.

Errors from possible energy lost:

Although extremely small sources of energy lost like the friction between the pivot and the beam of the catapult; the deflection of the beam; the sound energy generated upon impact and the dampening effect of the relatively soft grassy surface might have added up to form a considerable portion of the error, I remain to think that the largest contributor to energy lost was the air resistance from the rotation of the beam, projectile (before launch) and the launch dish.

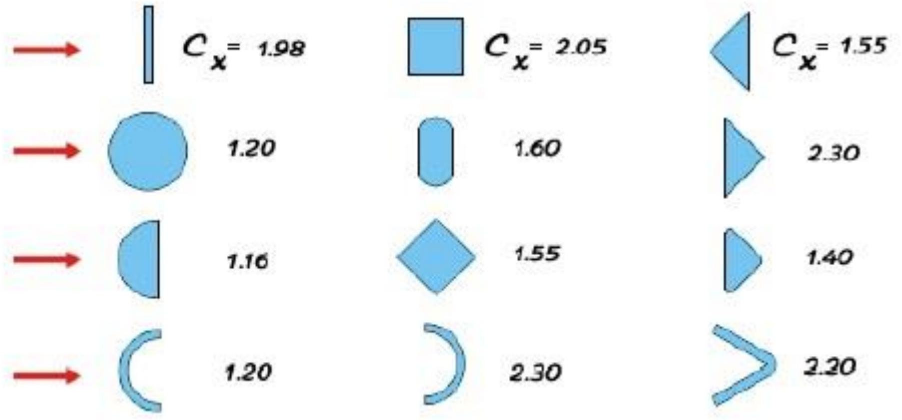


figure 4.1[3]

The reason to my assumption is that, as stated in the equation for aerodynamic drag ($F_D = \frac{1}{2}\rho v^2 C_D A$), we see that the force F_D due to aerodynamic drag is proportional to a half of the object's drag coefficient C_D (labelled C_x in figure 3.1) and its cross-sectional area A, which were both considerably larger than those of the spherical and relatively small projectile.

Finally, another area where energy might have been lost was found by some careful observations made by one of my group members who stated that the catapult was slightly tilted as it was set up on a small hill. Initially, this did not seem like a considerable source of error. However, as the slight tilt is magnified by the length of the catapult, it decreases the true height h_C of the counter mass relative to the centre of the earth. This can be significant as the gravitational potential energy of the counter mass is the only energy source to the system of the catapult.

In conclusion, what appeared to be a theoretically extremely simple task of launching a ball onto a target turned out to require way more knowledge and effort to tackle successfully. Analysing the effects due to simplifications and loss of energy in experimental conditions turned out to be significantly harder than the initial calculations made. This project was extremely rewarding and helped me to understand the difference between mathematicians, physics and engineers.

References:

[1] https://en.wikipedia.org/wiki/Drag_(physics)

[2] http://www.cems.uvm.edu/~tlakoba/15_spring/math_121/1997_AJP_ProjectMotionAirResist.pdf

[3] https://a2asimulations.com/forum/viewtopic.php?f=23&t=47896

[4] https://en.wikipedia.org/wiki/Drag_coefficient